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On the motion of sources of some Robinson–Trautman fields

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Abstract. We prove that if the source of a linearised Robinson–Trautman field is a simple pole particle in a background Minkowskian space-time, and if the motion of the particle is rectilinear in an inertial frame of reference, then (i) if the particle is uncharged it must move with uniform acceleration, or (ii) if the particle is charged it must move either with uniform acceleration or perform run-away motion from a state of uniform acceleration in the infinite past.

In the two preceding papers (Hogan and Imaeda 1979a, b) we studied in detail the Robinson–Trautman (1962) linearised fields of particles moving with uniform acceleration or performing run-away motion in a background Minkowskian space–time. *The purpose of this paper is to establish the uniqueness of these cases.* We shall prove that if the motion of a particle (of small† mass m and charge e) is rectilinear in an inertial frame of reference (IFR) with four-acceleration of magnitude $a(\sigma)$, where σ is proper-time along the world-line of the particle in the background Minkowskian space–time, and if the Robinson–Trautman field of the particle has a simple pole singularity (is devoid of 'directional' singularities), then the following conditions are satisfied:

(i) if $e = 0$ then $a = \text{constant}$;
(ii) if $e \neq 0$ then $a = a_0 + b_0 \exp(3m\sigma/2e^2) \qquad (-\infty < \sigma < \infty)$,

where a_0, b_0 are constants. Case (i) represents uniform acceleration. Case (ii) represents uniform acceleration if $b_0 = 0$, and run-away motion from a state of uniform acceleration in the infinite past if $b_0 \neq 0$.

To establish this result we begin with the following form for the line-element of a background Minkowskian space–time:

$$ds_0^2 = 2r^2 \underset{0}{P}^{-2}\, d\zeta\, d\bar{\zeta} - 2\, dr\, d\sigma - (1 - 2\underset{0}{H} r)\, d\sigma^2. \tag{1}$$

Here r, σ are real coordinates, while ζ, $\bar{\zeta}$ are complex (the bar indicating complex conjugation). Also

$$\underset{0}{P} = k_1(\tfrac{1}{2}\zeta\bar{\zeta} + k_2^2), \qquad \underset{0}{H} = \partial(\ln \underset{0}{P})/\partial\sigma = a(\sigma)\underset{0}{\xi}, \tag{2}$$

† We shall choose m and ϵ^2 to be small of first order, i.e. $m = O_1$, $e^2 = O_1$. Strictly speaking, in units for which $c = G = 1$, we have in mind that the dimensionless quantities ma, e^2a^2 are small of first order.

where k_1, k_2 are functions of σ with $k_1 k_2 = -1$, $a = \partial(\ln k_1)/\partial\sigma$ and

$$\underset{0}{\xi} = (\tfrac{1}{2}\zeta\bar{\zeta} - k_2^2)/(\tfrac{1}{2}\zeta\bar{\zeta} + k_2^2). \tag{3}$$

From (1) we deduce that the world-line $r = 0$ is time-like with σ proper-time along it. Let $\lambda^i = \lambda^i(\sigma)$, $i = 1, 2, 3, 4$, be the unit time-like tangent to $r = 0$, referred to the IFR. Let the particle move along the X^3 axis of the IFR so that $\lambda^1 = \lambda^2 = 0$. Then we may choose $k_1 = \lambda^3 + \lambda^4$ and $k_2 = \lambda^3 - \lambda^4$. If μ^i is the four-acceleration of the particle in the IFR, then

$$\mu^i \mu_i = a^2. \tag{4}$$

If k^i is tangent to the generators of the future-pointing null-cones with vertices on $r = 0$ (these have equations σ = constant), then we may write

$$\underset{0}{H} = -\mu_i k^i. \tag{5}$$

The line-element (1) is in Robinson–Trautman form, and we use it as the background space–time of a linearised charged Robinson–Trautman line-element thus,

$$ds^2 = 2r^2 P^{-2}\, d\zeta\, d\bar{\zeta} - 2\, dr\, d\sigma - h\, d\sigma^2 + O_2, \tag{6}$$

where

$$\begin{aligned} P &= \underset{0}{P}(1+Q), \qquad & h &= K - 2Hr - 2M/r + e^2/r^2, \\ K &= 1 + \underset{1}{K}, \qquad & H &= \underset{0}{H} + \underset{1}{H}, \\ M &= m + 2e^2 w, \qquad & w &= \underset{0}{w} + \underset{1}{w}. \end{aligned} \tag{7}$$

Here Q together with the quantities carrying a subscript 'one' are small of first order. The quantities K, H, M are functions of ζ, $\bar{\zeta}$, σ, and we take m and e to be constants. The four-potential is given by the one-form

$$\Phi = -e(1/r - w)\, d\sigma + O_2, \tag{8}$$

and $\underset{0}{w}$ (which appears in (7)) is equal to $\underset{0}{H}$. The quantities Q, $\underset{1}{K}$, $\underset{1}{H}$, $\underset{1}{w}$ are calculated from the linearised vacuum Einstein–Maxwell field equations. As an aid to solving them, one notices that the field must be rotationally symmetric about the X^3 axis of the IFR in the background Minkowskian space–time. This symmetry will be guaranteed in the solutions of the field equations if we assume that all functions depend on the variables ζ, $\bar{\zeta}$ in the combination $\zeta\bar{\zeta}$ (as in (2)). Instead of using $\zeta\bar{\zeta}$ as a new variable, we find that the field equations are greatly simplified if we use $\underset{0}{\xi}$ given by (3) as a new variable. The field equations can then be written (cf Hogan and Imaeda 1979b)

$$\begin{aligned} &\partial[(1-\underset{0}{\xi}^2)\partial \underset{1}{K}/\partial\underset{0}{\xi}]/\partial\underset{0}{\xi} = 12e^2a^2(1-3\underset{0}{\xi}^2) + 8e^2\dot{a}\underset{0}{\xi} - 12ma\underset{0}{\xi} + O_2, \\ &\partial[(1-\underset{0}{\xi}^2)\partial Q/\partial\underset{0}{\xi}]/\partial\underset{0}{\xi} + 2Q = \underset{1}{K}, \qquad \partial Q/\partial\sigma = \underset{1}{H}. \end{aligned} \tag{9}$$

We solve these equations by first solving for $\underset{1}{K}$, then for Q, and then for $\underset{1}{H}$. The tetrad components of the linearised Weyl and Maxwell tensors (in Newman–Penrose (1962) notation) are given on a suitable null-tetrad (cf Hogan and Imaeda 1979b) by

$$\psi_0 = 0, \qquad \psi_1 = 0,$$

$$\psi_2 = -(1/r^3)(m + 2e^2 a \underset{0}{\xi}) + e^2/r^4 + O_2,$$

$$\psi_3 = (1/2r^2)(\underset{0}{\bar{\zeta}}/P)[\partial \underset{1}{K}/\partial \underset{0}{\xi} - 6a(m + 2e^2 a \underset{0}{\xi})] + O_2,$$

$$\psi_4 = (1/r^2)(\underset{0}{\bar{\zeta}}^2/P^2)(\tfrac{1}{2}\partial^2 \underset{1}{K}/\partial \underset{0}{\xi}^2 - 6e^2a^2) + (1/r)(\underset{0}{\bar{\zeta}}^2/P^2)[2a\,\partial \underset{1}{K}/\partial \underset{0}{\xi}$$
$$- 6a^2(m + 2e^2 a \underset{0}{\xi}) - \partial(\partial \underset{1}{H}/\partial \underset{0}{\xi} + 2aQ)/\partial \underset{0}{\xi}] + O_2,$$

$$\Phi_0 = 0, \qquad \Phi_1 = -e/2r^2 + O_2, \qquad \Phi_2 = O_2. \tag{10}$$

We do not need to calculate $\underset{1}{w}$ in (7), as it does not appear in the linearised metric given by (6) and neither does it appear in the linearised Weyl or Maxwell tensors (10). When solving the linearised field equations (9), functions (of σ) of integration occur. To determine them we substitute the values of $\underset{1}{K}$, Q, $\underset{1}{H}$ into the tetrad components of the linearised Weyl tensor and find that, unless the functions of integration have certain values, the Weyl tensor will be singular not only on $r = 0$ (as one would expect of the field of a simple pole particle) but also when $\underset{0}{\xi} = \pm 1$. It follows from the second of (2) and equation (5) that a constant value of $\underset{0}{\xi}$, for each constant value of σ, corresponds to a choice of k^i, i.e. a generator of a future null-cone with vertex on $r = 0$. The singularities of the Weyl tensor which occur when $\underset{0}{\xi} = \pm 1$ are therefore 'directional' or 'wire' singularities. We have enough functions of integration at our disposal to remove these directional singularities from ψ_3. We are then left with ψ_4, which is given by

$$\psi_4 = (1/r)(\underset{0}{\bar{\zeta}}^2/P^2)\{a(C - 6ma + 6e^2\dot{a}) + [2\dot{\gamma} - \tfrac{1}{3}\dot{C}\underset{0}{\xi}(3 - \underset{0}{\xi}^2)](1 - \underset{0}{\xi}^2)^{-2}\} + O_2 \tag{11}$$

where the 'dot' indicates differentiation with respect to σ, $\gamma(\sigma)$ is a function of integration, and $C(\sigma)$ has the value

$$C = 6ma - 4e^2\dot{a}. \tag{12}$$

Expression (11) is singular on $r = 0$ and also when $\underset{0}{\xi} = \pm 1$ unless $\dot{\gamma} = 0$ and $\dot{C} = 0$. Thus C is constant, and we can solve for $a(\sigma)$ from (12) in the two cases $e = 0$ and $e \neq 0$ to obtain the results quoted at the beginning of this paper. The case $e = 0$ yields a result similar to that of Robinson and Robinson (1972), although it is not clear whether or not their assumptions coincide with ours.

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